MATH 2028 - Integration on bounded sets

So far, we have only talk about how to integrate bdd functions defined on a rectangle.

<u>GOAL:</u> Define the integral of f over a bdd subset $\Omega \subseteq \mathbb{R}^n$.

This can be done by a simple extension process. Let $f: \Omega \to \mathbb{R}$ be a bdd function defined on a bdd subset $\Omega \subseteq \mathbb{R}^n$. We can define its extension $\bar{f}: \mathbb{R}^n \to \mathbb{R}$ to a bdd function on the whole $\mathbb{R}^n$ by

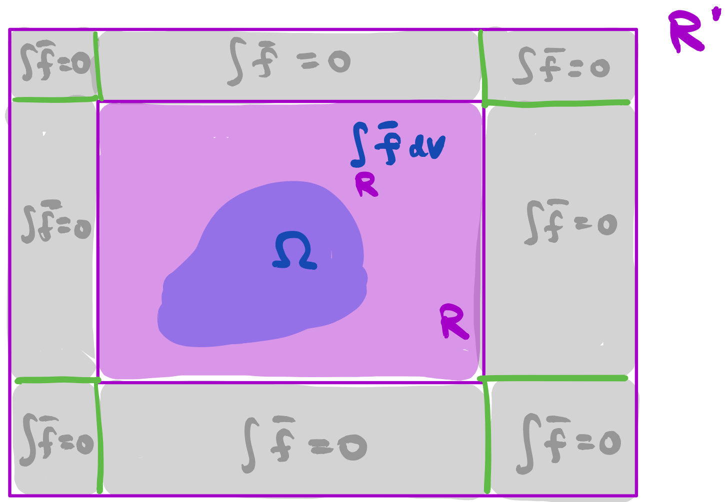
$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

Def^{n}: A bdd function $f: \Omega \to \mathbb{R}$ is integrable on a bdd subset $\Omega \subseteq \mathbb{R}^n$ if $\exists$ rectangle $R \supseteq \Omega$ s.t. the extension $\bar{f}$ is integrable on R. In this case, we define $\int_\Omega f\,dV = \int_R \bar{f}\,dV$.

Remark: The definition above seems to depend on the choice of the rectangle R containing Ω. The Lemma below makes the definition unambiguous.

Lemma: Suppose R and R' are two rectangles in $\mathbb{R}^n$ containing Ω. Then, $\bar{f}$ is integrable on R if and only if $\bar{f}$ is integrable on R'; moreover we have $\int_R \bar{f}\,dV = \int_{R'} \bar{f}\,dV$

Proof: It suffices to consider the case $R' \supseteq R \supseteq \Omega$. (Ex: why?) Since $\bar{f} \equiv 0$ outside Ω, the set of discontinuities of $\bar{f}$ is contained inside R and has measure zero iff $\bar{f}$ is integrable on R (or R').



The last assertion follows by a sub-division of R' into sub-rectangles as above.

$\blacksquare$

Recall that a continuous function $f : R \to \mathbb{R}$ on a rectangle R is always integrable. This is NOT always true for cts functions defined on a bdd subset $\Omega \subseteq \mathbb{R}^n$. But the situation is better when the boundary $\partial\Omega$ is not too wild.

Prop: Let $f : \Omega \to \mathbb{R}$ be a function.

Suppose (i) $\Omega \subseteq \mathbb{R}^n$ is a bdd subset whose boundary $\partial\Omega$ has measure zero (in $\mathbb{R}^n$)

(ii) f is continuous on Ω.

THEN, f is integrable on Ω.

Proof: Note that the set of discontinuities of the extension $\bar{f}$ is contained in $\partial\Omega$. The result follows from the integrability criteria. $\blacksquare$

Remark: Since the constant function $f(x) = 1, \forall x \in \Omega$ is continuous on Ω, if $\Omega \subseteq \mathbb{R}^n$ is a bdd subset with measure zero $\partial\Omega$, then we can define the volume of Ω to be

$$\boxed{\mathrm{Vol}(\Omega) := \int_\Omega 1 \, dV}$$

The following comparison result is often useful.

Prop: Let $f, g : \Omega \to \mathbb{R}$ be integrable functions on a bdd subset $\Omega \subseteq \mathbb{R}^n$ s.t. $\partial\Omega$ has measure zero. If $f(x) \leq g(x) \quad \forall x \in \Omega$, then

$$\boxed{\int_\Omega f \, dV \leq \int_\Omega g \, dV}$$

Proof: Exercise!